



SPECIAL THEORY OF RELATIVITY

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Postulates of Special Theory of Relativity. Lorentz Transformations. Minkowski space. The invariant interval, light cone and world lines. Space-time diagrams. Time-dilation, length contraction and twin paradox. Four-vectors: space-like, time-like and light-like. Four-velocity and acceleration. Four-momentum and energy-momentum relation. Concept of four-force. (20L)

INTRODUCTION

- Physics starts with the measurement of physical quantities and in mechanics the position and velocity are measured with respect of some reference (Frame of Reference) since there is nothing at absolute condition. Thus sense of relativity is very basic in physics.
- **Newtonian (Classical) Relativity:** It is assumed that Newton's laws of motion must be measured with respect to (relative to) some reference frame. According to Newton space and time are completely separable and time flows on uniformly regardless of any thing external. Here if light moves with velocity c and an observer moves with velocity u in the same direction, then the relative velocity of light will be $(c-u)$, but experiment shows the speed of light in vacuum in all direction is constant.
- **Einsteinian Relativity:** It is assumed space and time are not independent of each other.

FRAME OF REFERENCE

- **Inertial frame of reference:** A reference frame is called an **inertial frame** if it is not subjected to any external force, is observed to move in rectilinear motion at constant velocity. Newton laws are valid in this frame.
- **Non-inertial frame of reference:** Here one frame of reference moves non-uniformly or with acceleration with respect to other. Pseudo force comes into effect when the frame of reference start accelerates.

Inertial frame of reference
(including time axis)



Special Theory of Relativity

Non-inertial frame of reference



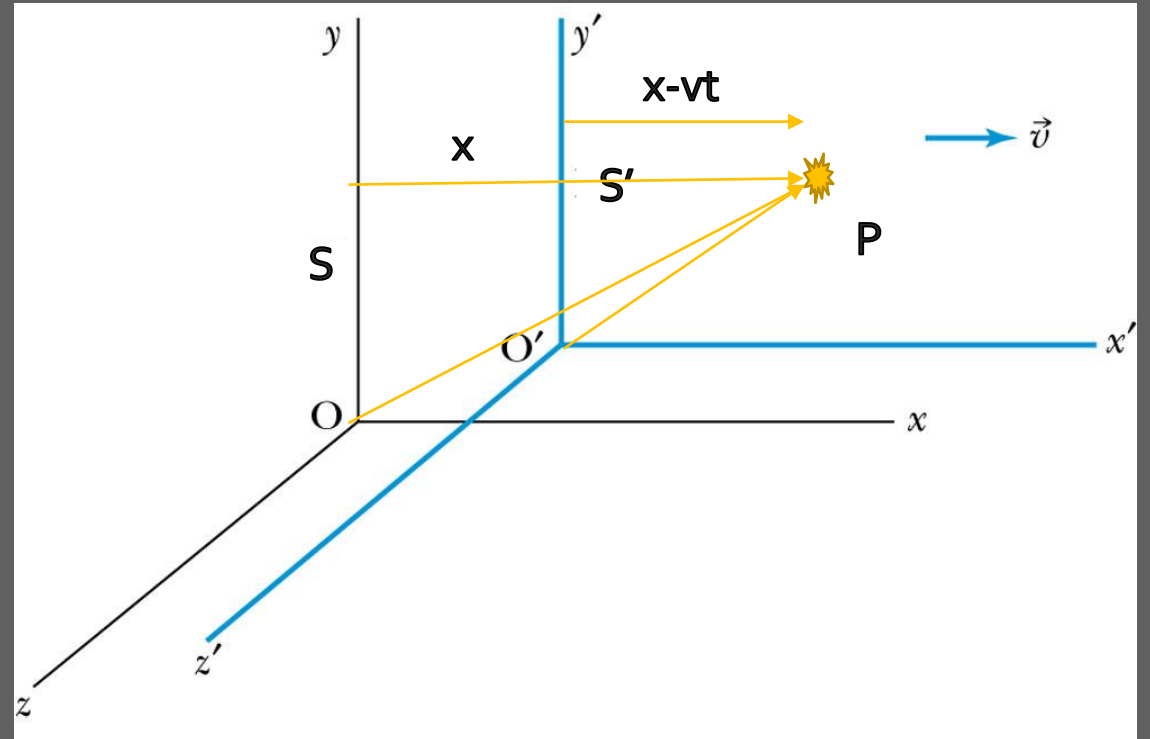
General Theory of Relativity

NEWTONIAN PRINCIPLE OF RELATIVITY

- **Newtonian principle of relativity** or **Galilean invariance** : If Newton's laws are valid in one reference frame, then they are also valid in another reference frame moving at a uniform velocity relative to the first system.

Galilean Transformation

- S is at rest and S' is moving with velocity $\vec{v}$
- Axes are parallel
- S and S' are said to be *INERTIAL COORDINATE SYSTEMS*
- For a point P
 - In system S: $P = (x, y, z, t)$
 - In system S': $P = (x', y', z', t')$



CONTD.....

- S' has a constant relative velocity (v) in the x-direction with respect to S

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t$$

- **Time** (t) for all observers is a *Fundamental invariant*, i.e., the same for all inertial observers.

- The relation between the velocity components in two frames are,

$$u_{x'} = u_x - v, \quad u_{y'} = u_y, \quad u_{z'} = u_z$$

A further differentiation w.r.t time gives the relation between acceleration components in two frames,

$$a_{x'} = a_x, \quad a_{y'} = a_y, \quad a_{z'} = a_z$$

- Considering a particle of mass m , the force in S frame

$$F = m(a_x + a_y + a_z) = m(a_{x'} + a_{y'} + a_{z'}) = F'$$

That is, Newton's 2nd law and hence all the laws of motion are invariant under Galilean transformation.

CONTD...

- It is to be noted that if there be a source of light at the origin of S frame, then velocity of light in the S' frame according to Galilean transformation will be,

$$c' = c - v$$

- Hence the speed of light is not constant i.e. Maxwell's equations did not hold ($c=1/\mu_0\epsilon_0$). Thus all laws of physics does not hold here.
- In 1905, Albert Einstein proposed a fundamental connection between space and time and that Newton's laws are only an approximation.

CONCEPT OF ETHER

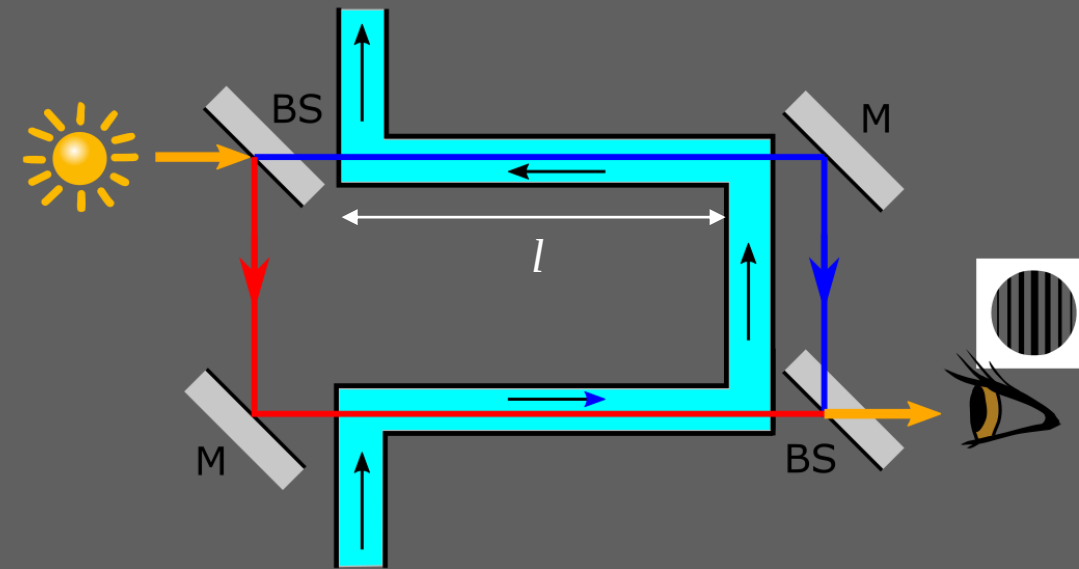
- The wave nature of light suggested that there existed a propagation medium called the *luminiferous ether* or just **ether**.
- Ether had to have such a low density that the planets could move through it without loss of energy
- It also had to have an elasticity to support the high velocity of light waves
- Ether was proposed as **an absolute reference system** in which the speed of light was this constant and from which other measurements could be made.

FIZEAU'S EXPERIMENT

The **Fizeau experiment** was carried out by [Hippolyte Fizeau](#) in 1851 to measure the relative speeds of light in moving water. Figure shows a highly simplified representation of the Fizeau experiment. The incident light is split by a beam splitter (BS) into two beams that pass through two water columns flowing in opposite directions. The two beams are then recombined to form an interference pattern that can be observed and interpreted. Assuming the water flows in the pipes at a speed v , according to the non-relativistic theory of luminiferous ether, the speed of light should be either increased or decreased depending on the direction as it is "dragged" through the ether framework by the water.

According to Stokes' complete ether drag hypothesis, the total speed of the light beams should be the simple sum of their speed through the water and the speed of the water. In other words, if n is the refractive index of water and (c/n) is the speed of light in stationary water, then the predicted speed of light w in one arm will be

$$w_+ = (c/n) + v$$



FIZEAU'S EXPERIMENT

the predicted speed in the other arm is

$$w_+ = (c/n) + v$$

The time difference between the two paths traversed by the light is,

$$\Delta t = l \left[\frac{1}{\frac{c}{n} - v} - \frac{1}{\frac{c}{n} + v} \right] = \frac{2lvn^2}{c^2} \left(1 - \frac{v^2 n^2}{c^2} \right)^{-1} = \frac{2lvn^2}{c^2} \quad (\text{when } vn \ll c)$$

Due to this time difference there is a shift in interference fringe, $\Delta s = v \Delta t = \frac{2lvn^2 v}{c^2} = \frac{2lvn^2}{c\lambda}$

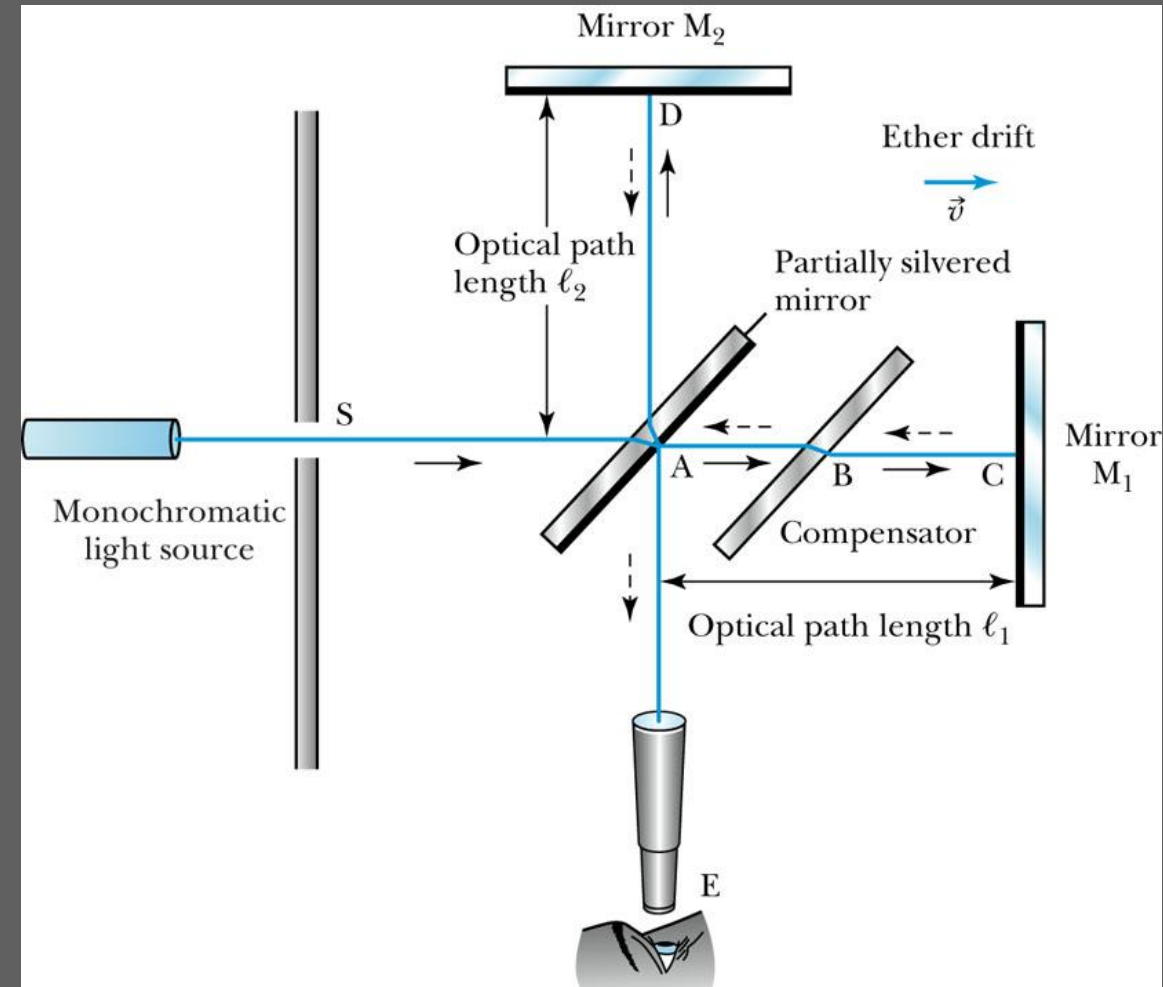
Where, ν = frequency and λ = wavelength. But in experiment, shift was found to be $\Delta s = f \frac{2lvn^2}{c\lambda}$

And $f = (1 - \frac{1}{n^2})$ the dragging co-efficient.

The conclusion that the velocities were not simply additive. Einstein wrote that the Fizeau experiment “decides in favour of [the velocity addition law] derived from the theory of relativity, and the agreement is, indeed, very exact.

THE MICHELSON-MORLEY EXPERIMENT

- The Michelson-Morley experiment was an attempt to show the existence of ether.
- Albert Michelson (1852–1931) was the first U.S. citizen to receive the Nobel Prize for Physics (1907), and built an extremely precise device called an *interferometer* to measure the minute phase difference between two light waves traveling in mutually orthogonal directions.
- 1. AC is parallel to the motion of the Earth inducing an “ether wind”
 2. Light from source S is split by mirror A and travels to mirrors C and D in mutually perpendicular directions
 3. After reflection the beams recombine at A slightly out of phase due to the “ether wind” as viewed by telescope E.



THE MICHELSON-MORLEY EXPERIMENT

Assuming the Galilean Transformation

Time t_1 from A to C and back:

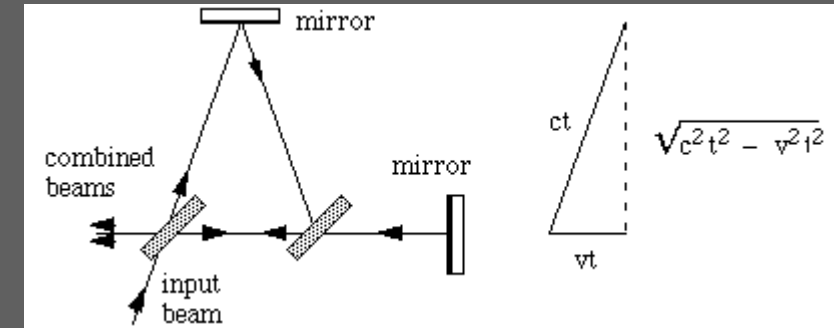
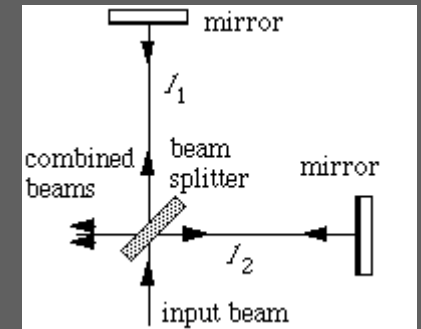
$$t_1 = \frac{\ell_1}{c+v} + \frac{\ell_1}{c-v} = \frac{2c\ell_1}{c^2 - v^2} = \frac{2\ell_1}{c} \left(\frac{1}{1 - v^2/c^2} \right)$$

Time t_2 from A to D and back:

$$t_2 = \frac{2\ell_2}{\sqrt{c^2 - v^2}} = \frac{2\ell_2}{c} \frac{1}{\sqrt{1 - v^2/c^2}}$$

So that the change in time is:

$$\Delta t = t_2 - t_1 = \frac{2}{c} \left(\frac{\ell_2}{\sqrt{1 - v^2/c^2}} - \frac{\ell_1}{1 - v^2/c^2} \right)$$



$$\Delta t = \frac{lv^2}{c^3} \quad l_1 = l_2 = l$$

Using Binomial Expansion

THE MICHELSON-MORLEY EXPERIMENT

Upon rotating the apparatus, the optical path lengths ℓ_1 and ℓ_2 are interchanged producing a different change in time: (note the change in denominators)

$$\Delta t' = t'_2 - t'_1 = \frac{2}{c} \left(\frac{\ell_2}{1 - v^2/c^2} - \frac{\ell_1}{\sqrt{1 - v^2/c^2}} \right) \xrightarrow{\quad} \begin{matrix} \ell_1 = \ell_2 = l \\ \Delta t' = -\frac{lv^2}{c^3} \end{matrix}$$

Using Binomial Expansion

$$\Delta t - \Delta t' = \frac{2lv^2}{c^3}$$

Therefore shift in fringes = $v(\Delta t - \Delta t') = \frac{2l}{\lambda} \left(\frac{v}{c}\right)^2$, Using the Earth's orbital speed as: $V = 3 \times 10^4$ m/s, together with $\ell_1 \approx \ell_2 = 1.2$ m and $\lambda = 6000$ Å, then shift = 0.04 which can be measured easily in the experiment.

THE MICHELSON-MORLEY EXPERIMENT

CONCLUSION:

- Michelson noted that he should be able to detect a phase shift of light due to the time difference between path lengths but found none.
- He thus concluded that the hypothesis of the stationary ether must be incorrect.
- After several repeats and refinements with assistance from Edward Morley (1893-1923), again *a null result*.
- ***Thus, ether does not seem to exist!***

EINSTEIN'S POSTULATES

- Albert Einstein (1879–1955) was only two years old when Michelson reported his first null measurement for the existence of the ether.
- At the age of 16 Einstein began thinking about the form of Maxwell's equations in moving inertial systems.
- In 1905, at the age of 26, he published his startling proposal about the **principle of relativity**, which he believed to be fundamental.

With the belief that Maxwell's equations must be valid in all inertial frames, Einstein proposes the following postulates:

- 1) **The principle of relativity:** The laws of physics are the same in all inertial systems. There is no way to detect absolute motion, and no preferred inertial system exists.
- 2) **The constancy of the speed of light:** Observers in all inertial systems measure the same value for the speed of light in a vacuum.

THE PROBLEM OF SIMULTANEITY

- In Newtonian physics we previously assumed that $t = t'$
 - Thus with "synchronized" clocks, events in S and S' can be considered simultaneous
- Einstein realized that each system must have its own observers with their own clocks and meter sticks
 - Thus events considered simultaneous in S may not be in S'

LORENTZ TRANSFORMATIONS

The special set of linear transformations that:

1) preserve the constancy of the speed of light (c) between inertial observers;
and,

2) account for the problem of simultaneity between these observers

known as the **Lorentz transformation equations**

LORENTZ TRANSFORMATIONS

To derive the Lorentz Transformation equations, we consider two observers at two inertial frame, moving with respect to each other at a velocity v along the x -direction. We assume that the two observers have synchronized their clocks so that $t = t' = 0$, and at this point in time, the two origins coincide. According to Galilean transformation,

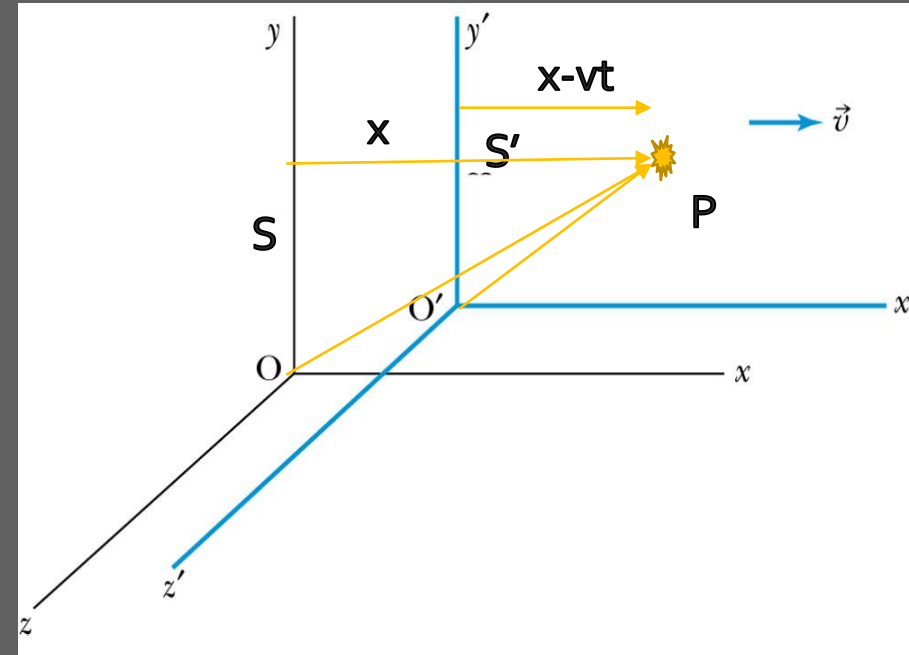
$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t \quad (1)$$

We find G.T violates the postulates of STR. So we assume,

$$x' = \alpha(x - vt) \quad (2)$$

Where, α is a constant. Equation (2) has been assumed on the following basis,

1. The relation between x' and x should be linear, so that a single event in S corresponds to single event in S' .
2. At very low value of v , $\alpha=1$ and we have G.T.



LORENTZ TRANSFORMATIONS

The inverse transformation of equation (2) can be written as,

$$x = \alpha'(x' + vt')$$
(3)

Second and third part of equation (1) remain same but last part will be in applicable since t and t' are not equal. Putting the value of x' in (2) in equation (3), we have

$$x = \alpha\alpha'(x - vt) + \alpha'vt'$$
$$t' = \alpha t + \left(\frac{1 - \alpha\alpha'}{\alpha'v}\right)x$$
(4)

Now suppose that at $t=t'=0$, a signal of light is given out from common origin of S and S' . Then the signal will propagate in the two systems such that, $x = ct$

(5)

$$x' = ct'$$
(6)

Putting the x' value from (2) and t' value from (4) in (6), we have

$$\alpha(x - vt) = c\alpha t + \left(\frac{1 - \alpha\alpha'}{\alpha'v}\right)cx \quad \longrightarrow \quad x = ct \left[\frac{1 + \frac{v}{c}}{1 - \left(\frac{1}{\alpha\alpha'} - 1\right)(c/v)} \right]$$
(7)

LORENTZ TRANSFORMATIONS

- Comparing (5) and (7), $1 + \frac{v}{c} = 1 - (\frac{1}{\alpha\alpha'} - 1)(c/v)$

$$\sqrt{\alpha\alpha'} = (1 - \frac{v^2}{c^2})^{-1/2} \quad (8)$$

$\alpha\alpha'$ depends on v^2 , so not affected by the sign of v . so we choose,

$$\alpha = \alpha' = (1 - \frac{v^2}{c^2})^{-1/2} \quad (9)$$

Putting the value of α we get the Lorentz Transformation equations,

$$x' = \frac{x-vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t-\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad (10)$$

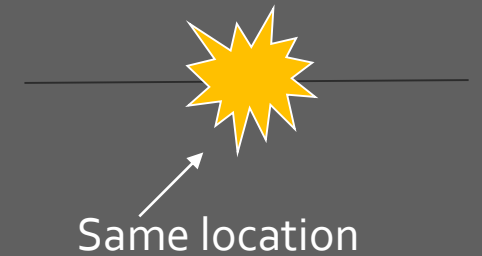
The inverse Lorentz Transformation equations are,

$$x = \frac{x'+vt}{\sqrt{1-\frac{v^2}{c^2}}}, \quad y = y', \quad z = z', \quad t = \frac{t'+\frac{vx}{c^2}}{\sqrt{1-\frac{v^2}{c^2}}} \quad (11)$$

CONSEQUENCES OF THE LORENTZ TRANSFORMATION:

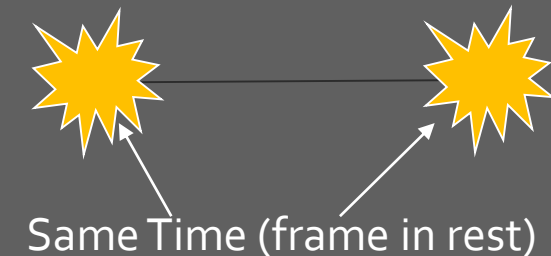
To understand *time dilation* the idea of **proper time** must be understood:

- The term **proper time**, T_0 is the time difference between two events occurring at the same position in a system as measured by a clock at that position.



To understand *length contraction* the idea of **proper length** must be understood:

- Let an observer in each system S and S' have a meter stick at rest in their own system such that each measure the same length at rest. The length as measured at rest is called the **proper length**.



CONSEQUENCES OF THE LORENTZ TRANSFORMATION:

- **Time Dilation**(Clocks in S' run slow with respect to stationary clocks in S):

Consider two observers, A and B, moving relative to each other with a constant velocity v . Observer A remains stationary in their reference frame (S), while observer B moves with velocity v in a parallel direction in their own reference frame S' . Let's say observer A observes two events, A_1 and A_2 , occurring at the same location in their frame but at different times i.e. $x_2 - x_1 = 0$.

Similarly, observer B observes the same events, B_1 and B_2 , in their frame. Because of the relative motion between S and S' , time appears to flow differently for observers A and B.

Let t_1 and t_2 be the time intervals between events A_1 and A_2 as measured by observer A in frame S . Similarly, let t_1' and t_2' be the time intervals between events B_1 and B_2 as measured by observer B in frame S' .

The Lorentz transformation equations relate the time measured in each frame:

$$t_1' = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad t_2' = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

TIME DILATION, CONTD...

The time interval in S' frame is,

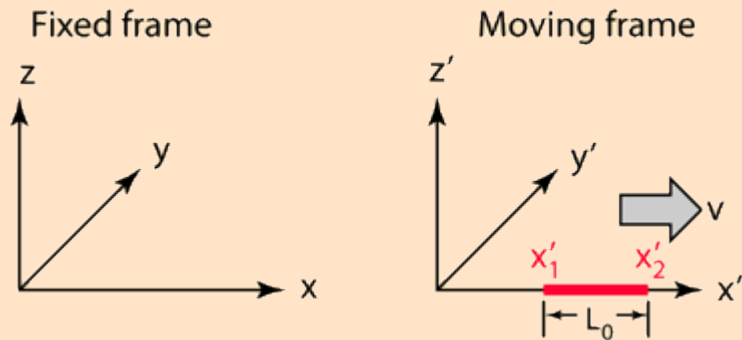
$$t'_2 - t'_1 = \frac{(t_2 - t_1) - (v/c^2)(x_2 - x_1)}{\sqrt{1 - v^2/c^2}}$$

Note here . $x_2 - x_1 = 0$ in S with a proper time: $T_0 = t_2 - t_1$

$$T' = \frac{T_0}{\sqrt{1 - v^2/c^2}} = \gamma T_0$$

- 1) $T' > T_0$ or the time measured between two events at *different positions* in S' frame is greater than the time between the same events at *one position in S frame*: **time dilation**.
- 2) The events do not occur at the same space and time coordinates in the two system.

CONSEQUENCES OF THE LORENTZ TRANSFORMATION:



The length of any object in a moving frame will appear foreshortened in the direction of motion, or contracted. The amount of contraction can be calculated from the [Lorentz transformation](#). The length is maximum in the frame in which the object is at rest.

If the length $L_0 = x'_2 - x'_1$ is measured in the moving reference frame, then $L = x_2 - x_1$ can be calculated using the Lorentz transformation.

$$L_0 = x'_2 - x'_1 = \frac{x_2 - vt_2 - x_1 + vt_1}{\sqrt{1 - \frac{v^2}{c^2}}}$$

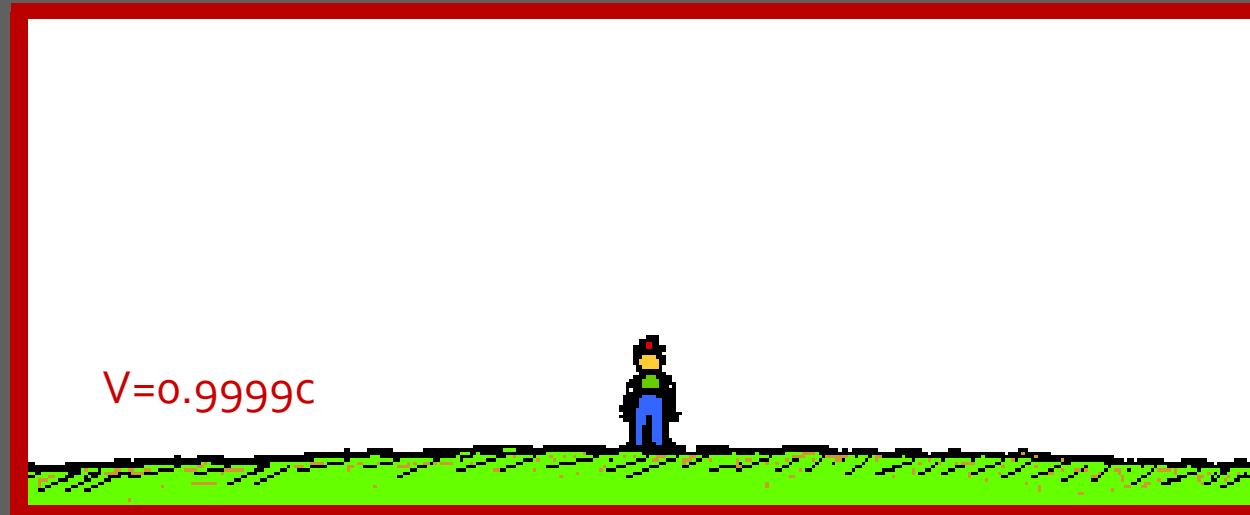
But since the two measurements made in the fixed frame are made simultaneously in that frame, $t_2 = t_1$, and

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} = \frac{L_0}{\gamma}$$

Length contraction

CONCLUSION: This equation shows that the length L' measured by observer B is smaller than the proper length L measured by observer A. In other words, the length of the rod appears contracted along the direction of motion as observed by observer B. This phenomenon, known as length contraction, is a fundamental aspect of special relativity. This phenomena is known as **Lorentz-FitzGerald contraction**.

MOVING OBJECTS APPEAR SHORTER (ANIMATION)



RELATIVITY OF SIMULTANEITY

- If two events occur at same time in a particular frame of reference at two different point i.e. $x_2 - x_1 \neq 0$, they are said to be simultaneous. Let two events occur simultaneously in S frame i.e. $(t_2 - t_1) = 0$. The time recorded of occurring two events from S' frame is obtained by lorentz transformation as follows,

$$t'_1 = \frac{t_1 - \frac{vx_1}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$t'_2 = \frac{t_2 - \frac{vx_2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$\text{Thus, } (t'_2 - t'_1) = (-\frac{v}{c^2} (x_2 - x_1)) / \sqrt{1 - \frac{v^2}{c^2}} = (\frac{v}{c^2} (x_1 - x_2)) / \sqrt{1 - \frac{v^2}{c^2}}$$

This indicates two events which are simultaneous in S frame are not simultaneous in S' frame i.e. **Simultaneity is not absolute but relative.**

TWIN PARADOX

- The Twin Paradox is one of the most fascinating concepts in the realm of special relativity. It's a scenario where one twin, let's call them Alice, embarks on a journey through space at a high speed while the other twin, Bob, remains on Earth. When Alice returns, she finds that less time has passed for her compared to Bob, even though she experienced time dilation during her journey. At first glance, it seems paradoxical because both twins should experience time passing at the same rate according to classical intuition. However, according to Einstein's theory of special relativity, time is relative and can be experienced differently by observers moving at different speeds relative to each other.
- The key to understanding the Twin Paradox lies in the concept of time dilation. When an object moves at a significant fraction of the speed of light, time appears to pass slower for that object relative to a stationary observer. This effect becomes more pronounced as the speed approaches the speed of light. In the case of the traveling twin, Alice, because she is moving at a high speed relative to Bob, her experience of time is slowed down compared to Bob's.
- From Alice's perspective, it is Bob who is moving away from her and then coming back, so she could argue that it's Bob who should be younger when they reunite. However, this is where the asymmetry arises: Alice experiences acceleration when she turns around to come back to Earth, which breaks the symmetry between the twins. This acceleration introduces another factor, known as the "twin that accelerates ages less" rule, which explains why Alice ends up younger.
- In essence, the Twin Paradox is not really a paradox in the sense of a logical contradiction, but rather a thought experiment that highlights the counterintuitive nature of time as described by special relativity. It demonstrates that time is not an absolute quantity but is instead relative and can be affected by motion and gravity.

COMPLEX MINKOWSKI SPACE

In Four-dimensional complex Minkowski space the Fourth Coordinate be $x_4 = ict$. The Minkowski space is a complex four-dimensional space (World space). The Lorentz transformation can be visualized as a rotation in the complex Minkowski space that preserves the length of the vector. $x'^2 + y'^2 + z'^2 - c^2 t'^2 = x^2 + y^2 + z^2 - c^2 t^2$. The length of the vector $(x^2 + y^2 + z^2 - c^2 t^2)^{1/2}$, in this case may be real or imaginary, real if $x^2 + y^2 + z^2 - c^2 t^2$ is positive, or imaginary if $x^2 + y^2 + z^2 - c^2 t^2$ is negative. This means that the Lorentz transformation is an orthogonal transformation in the four-dimensional complex Minkowski space.

A physical event in Minkowski space is described by a point with co-ordinate (x, y, z, ict) . This is known as **world point**. The path of a particle in this space is known as **world line**.

LORENTZ TRANSFORMATION : A ROTATION IN COMPLEX MINKOWSKI SPACE

- We use the symbols, $x_1 = x, x_2 = y, x_3 = z, x_4 = ict$
- We consider a rotation in $x_1 - x_4$ plane through an angle θ (Figure). The co-ordinate (x_1, x_4) of a point P are related to the co-ordinate (x_1', x_4') by the relation, $\begin{bmatrix} x_1 \\ x_4' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}$ and inverse relation is,

$$\begin{bmatrix} x_1 \\ x_4 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_1' \\ x_4' \end{bmatrix}$$
- If $x_1' = 0$, then, $x_1 = -\sin\theta \cdot x_4'$ and $x_4 = \cos\theta \cdot x_4'$ which gives, $\frac{x_1}{x_4} = -\tan\theta$, inversely, $\tan\theta = -\frac{x_1}{x_4} = -\frac{x}{ict} = \frac{iv}{c}$ where, $\frac{x}{t} = v$
- Consider the second figure, $\sin\theta = \frac{\frac{iv}{c}}{\sqrt{1-\frac{v^2}{c^2}}}$ and $\cos\theta = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$

CONTD.....

- Hence, putting the value of $\sin\theta$ and $\cos\theta$ we have,

$$x'_1 = x' = \frac{x_1 + i\frac{v}{c}x_4}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x - vt}{\sqrt{1 - \frac{v^2}{c^2}}}$$

$$x'_2 = y' = y$$

$$x'_3 = z' = z$$

$$x'_4 = ict' = \frac{-\frac{iv}{c}x_1 + x_4}{\sqrt{1 - \frac{v^2}{c^2}}} = ic \left(\frac{t - \frac{vx}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} \right)$$

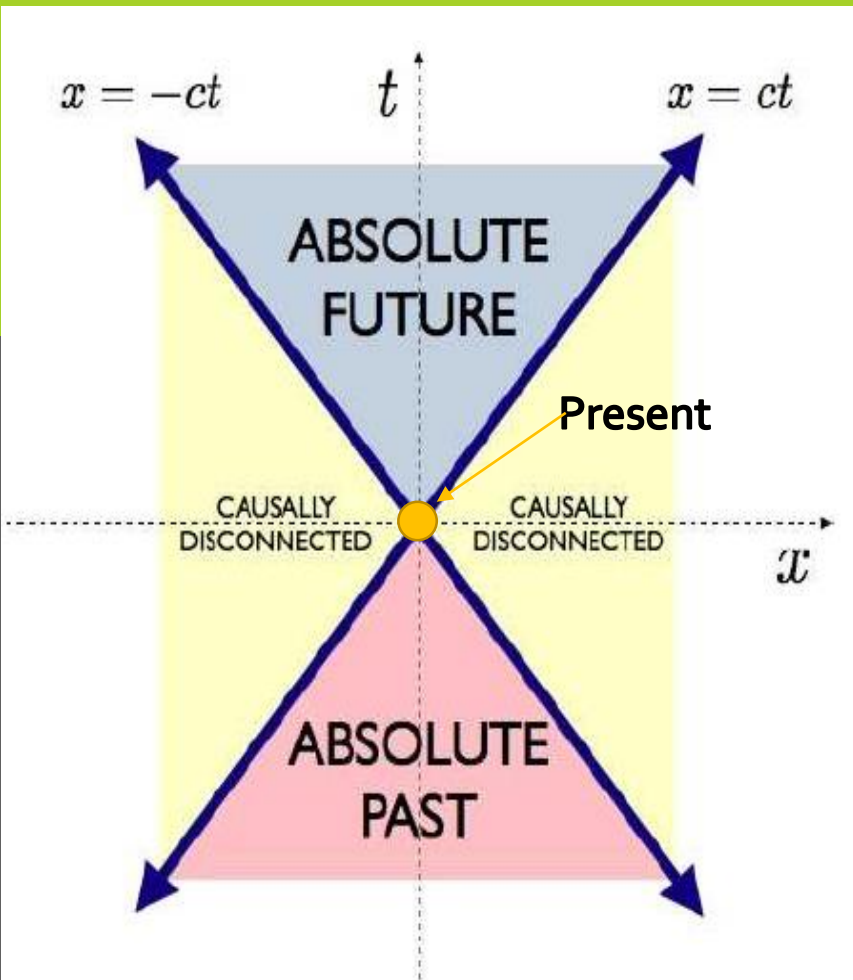
Which are the relation as obtained in Lorentz transformation.

SPACE TIME INTERVAL

- It represents the distance between two events in space and time, taking into account the relativistic effects of both space and time. If the two events are occurred at $P_1(x_1, y_1, z_1, t_1)$ and $P_2(x_2, y_2, z_2, t_2)$ then in special relativity, the space-time interval between these two events is defined as:

$$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$$

- Where: Δs is the space-time interval, c is the speed of light in a vacuum, $\Delta t = (t_2 - t_1)$ is the time difference between the events, $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, and $\Delta z = z_2 - z_1$ are the differences in spatial coordinates.
- In this formulation, the space-time interval is invariant, meaning it has the same value in all inertial frames of reference. This invariance is a key principle of special relativity.

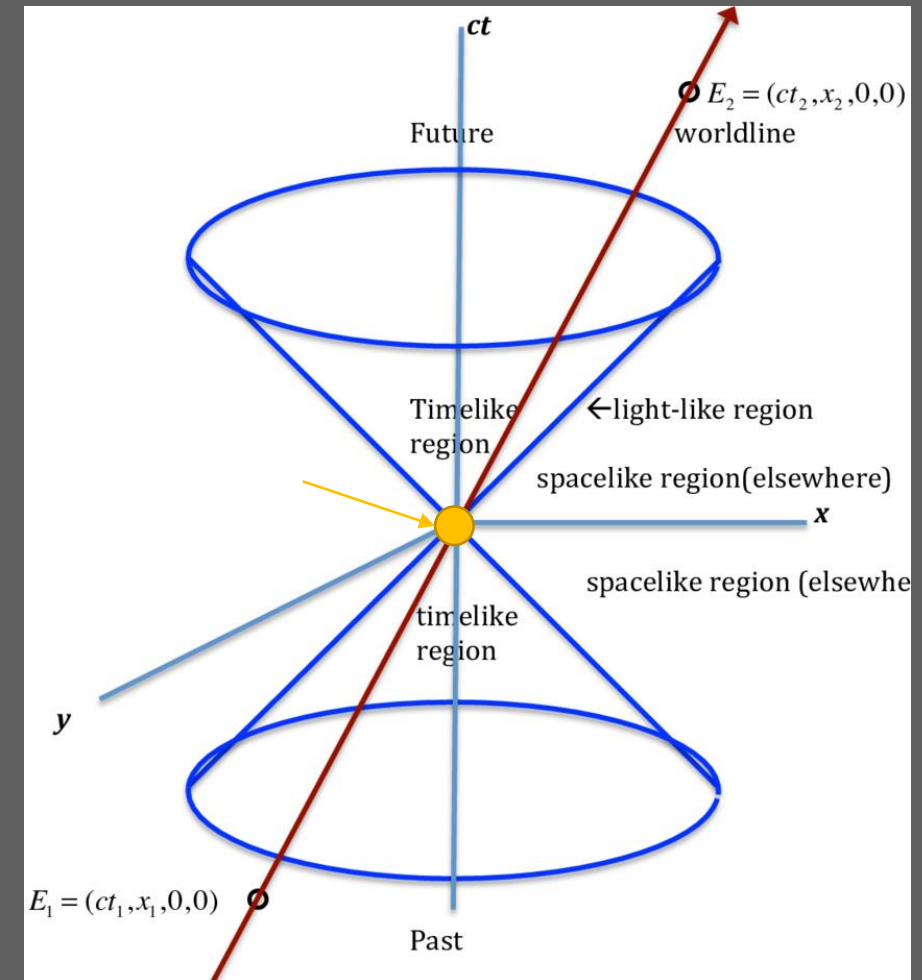


$\Delta s^2 = c^2 \Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$, $\Delta s^2 = c^2 \Delta t^2 - \Delta d^2$, Where,
 $\Delta d^2 = \Delta x^2 + \Delta y^2 + \Delta z^2$
 $\Delta s^2 = 0$, is a critical value. At this value $\Delta d / \Delta t = c$, i.e. observer moves in light velocity. So it is lightlike surface.

$\Delta s^2 > 0$ means, $\Delta d > 0$ (positive) for $\Delta t = 0$, So it is called space like. Here, $\Delta d / \Delta t > c$, i.e. observer has to move faster than light to separate two events which is impracticable.

▪ $\Delta s^2 < 0$ means, $\Delta t > 0$ (positive) for $\Delta d = 0$, So it is called time like.

Here, $\Delta d / \Delta t < c$, i.e. observer has less speed than light to separate two events which is practicle.



FOUR VECTOR MECHANICS:

Position four-vector :

A position four-vector is a mathematical construct used in physics, particularly in the theory of relativity, to describe the location of an event in space-time. It consists of four components: three spatial components and one temporal component. The position four-vector is denoted as x_μ , where the subscript μ ranges from 0 to 3, representing the time component (0) and the three spatial components (1, 2, 3).

- Mathematically, a position four-vector is represented as:

$$x_\mu = (ct, x, y, z)$$

- Where: ct represents the time component multiplied by the speed of light (c) to maintain consistent units.
- x, y, z represent the spatial components.
- This four-dimensional representation allows for a unified treatment of space and time, enabling physicists to analyze events in a way that accounts for the relativistic effects of both spatial and temporal dimensions.

FOUR VECTOR MECHANICS:

Proper length interval:

In four-vector mechanics, the proper length interval, also known as the proper distance or proper time interval, is a fundamental concept used to measure distances or time intervals between events in space-time while accounting for relativistic effects such as time dilation and length contraction. It is calculated using the space-time interval formula, which is based on the Minkowski metric:

$$ds^2 = c^2 dt^2 - (dx^2 + dy^2 + dz^2)$$

Where: dt is the time interval between the events as measured by the observer at rest relative to the events and dx, dy, dz are the spatial intervals between the events in the observer's frame of reference.

This proper length interval is an invariant quantity. The time interval dt is not an invariant quantity in four vector mechanics, so we define $d\tau$ as, $d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2}(dx^2 + dy^2 + dz^2)$, which is invariant quantity under transformation. This is known as proper time.

FOUR VECTOR MECHANICS:

$$\cdot d\tau^2 = \frac{ds^2}{c^2} = dt^2 - \frac{1}{c^2} (dx^2 + dy^2 + dz^2)$$

$$\text{Or, } d\tau^2 = dt^2 \left[1 - \frac{1}{c^2} \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\} \right]$$

$$\cdot \text{Or, } \frac{d\tau}{dt} = \sqrt{1 - \frac{v^2}{c^2}} \text{ where, } v^2 = \left\{ \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 \right\}$$

FOUR VECTOR MECHANICS:

Velocity four-vector

- To find the velocity four-vector u_μ , which represents the velocity of the object in space-time, we differentiate the position four-vector with respect to the proper time τ , which is the time measured by an observer moving with the object.

$$u_\mu = dx_\mu / d\tau$$

- Differentiating each component of x_μ with respect to τ , we obtain:

For the time component ict : $d(ict)/d\tau = ic(dt/d\tau) = ic\gamma$

For the spatial components x, y, z :

$$d(x)/d\tau = (dx/dt)(dt/d\tau) = \gamma v_x, \quad d(y)/d\tau = (dy/dt)(dt/d\tau) = \gamma v_y, \quad d(z)/d\tau = (dz/dt)(dt/d\tau) = \gamma v_z$$

- Where v_x, v_y, v_z are the components of the velocity vector v relative to an observer and γ is the Lorentz factor. Thus, the velocity four-vector u_μ is: $u_\mu = (c\gamma, \gamma v_x, \gamma v_y, \gamma v_z)$
- This represents the velocity of an object in space-time, accounting for both spatial and temporal components considering the effects of special relativity.

FOUR VECTOR MECHANICS:

- Norm of velocity four vector:
$$\begin{aligned}u_{\mu} u_{\mu} &= c^2 \gamma^2 - \gamma^2 (v_x^2 + v_y^2 + v_z^2) \\&= c^2 \gamma^2 - \gamma^2 v^2 \\&= (c^2 - v^2) \gamma^2 \\&= c^2\end{aligned}$$

Four momentum Vector:

To derive the four-momentum vector P_{μ} , we need to multiply each component of the four-velocity vector by the rest mass m_0 of the object. So, the components of the four-momentum vector are:

Temporal component P_0 : $P_0 = m_0 u_0 = m_0 c \gamma = i m c = i E / c$

Spatial components P_i : $P_i = m_0 u_i = m_0 \gamma v_i = m v_i$

where v_i represents the spatial components of the velocity.

FOUR VECTOR MECHANICS:

- Norm of velocity four vector:
$$\begin{aligned} P_\mu P_\mu &= -E^2/c^2 + m_0^2 \gamma^2 (v_x^2 + v_y^2 + v_z^2) \\ &= -E^2/c^2 + m_0^2 \gamma^2 v^2 \\ &= m_0^2 \gamma^2 [-E^2/m_0^2 \gamma^2 c^2 + v^2] \\ &= m_0^2 \gamma^2 [-E^2/m^2 c^2 + v^2] \\ &= m_0^2 \gamma^2 [-E^2 c^2/E^2 + v^2] \\ &= m_0^2 \gamma^2 [-c^2 + v^2] \\ &= m_0^2 c^2 \end{aligned}$$

FOUR VECTOR MECHANICS:

Four-acceleration vector:

- The four-acceleration vector A_μ is the derivative of the four-velocity with respect to proper time τ . It's given by:

$$A_\mu = dU_\mu/d\tau = (dU_\mu/dt) \cdot (dt/d\tau) = \gamma(dU_\mu/dt)$$

- Let's break down each component:

Temporal component A_0 : $A_0 = \gamma(dU_0/dt) = \gamma d(ic\gamma)/dt = ic\gamma (d\gamma/dt)$

Spatial components A_i : $A_i = \gamma(dU_i/dt) = \gamma(d(\gamma v_i)/dt) = \gamma[\gamma dv_i/dt + v_i d\gamma/dt]$

- where i ranges over the spatial dimensions.

FOUR VECTOR MECHANICS:

- To derive the components of the four-force vector F_μ from the four-acceleration vector A_μ , we'll use the equation of motion in special relativity:

$$F_\mu = m_0 A_\mu = m_0 (dU_\mu/d\tau) = dP_\mu/d\tau = \gamma(dP_\mu/dt)$$

- where m_0 is the rest mass of the object.
- Let's break down each component:

Temporal component F_0 : $F_0 = \gamma(dP_0/dt) = \gamma(d(iE/c)/dt) = (i\gamma/c)dE/dt$

Spatial components F_i : $F_i = \gamma(dP_i/dt)$

- where i ranges over the spatial dimensions.

FOUR VECTOR MECHANICS:

- **Conservation Law**

The conservation of four-momentum states that in any isolated system, the total four-momentum before an interaction is equal to the total four-momentum after the interaction. Mathematically, if we consider two particles with four-momenta P_1 and P_2 interacting and resulting in particles with four-momenta P_1' and P_2' , then:

$$(P_1)_\mu + (P_2)_\mu = (P_1')_\mu + (P_2')_\mu$$

This equation must hold for each component of the four-momentum.

SOLVED PROBLEM:

SHOW THAT FOUR VELOCITY AND FOUR FORCE VECTORS ARE ORTHOGONAL.

$$U^\mu F_\mu = m \left(U^\mu \frac{dU_\mu}{d\tau} \right)$$

The norm of the four-velocity is constant and equal to c^2 :

$$U^\mu U_\mu = \gamma^2 (c^2 - v^2) = c^2$$

This implies:

$$\frac{d}{d\tau} (U^\mu U_\mu) = \frac{d}{d\tau} (c^2) = 0$$

Applying the product rule to the derivative:

$$\frac{d}{d\tau} (U^\mu U_\mu) = 2U^\mu \frac{dU_\mu}{d\tau}$$

Since the derivative of a constant (in this case, c^2) is zero:

$$2U^\mu \frac{dU_\mu}{d\tau} = 0$$

Thus:

$$U^\mu \frac{dU_\mu}{d\tau} = 0$$

Therefore:

$$U^\mu F_\mu = m \left(U^\mu \frac{dU_\mu}{d\tau} \right) = m \cdot 0 = 0$$

Conclusion

We have shown that the inner product $U^\mu F_\mu$ is zero, which means that the four-velocity U^μ and the four-force F^μ are orthogonal.

SOLVED PROBLEM:

EXAMPLE 15.13 Show that if l_0^3 is the rest volume of a cube, then

$$l_0^3 (1 - \beta^2)^{1/2}$$

is the volume as viewed from a reference frame moving with a uniform velocity v parallel to an edge of the cube.

Solution. The length of the cube at rest $= l_0$.

Hence, the volume at rest

$$V_0 = l_0^3 \quad (1)$$

Along the direction of motion parallel to an edge, the length l_0 is viewed as

$$= l_0 (1 - \beta^2)^{1/2} \quad (2)$$

In the other two directions, there is no Lorentz contraction. Hence, the volume of the moving cube

$$= l_0 \times l_0 \times l_0 (1 - \beta^2)^{1/2} = l_0^3 (1 - \beta^2)^{1/2} \quad (3)$$

where $\beta = v/c$.

SOLVED PROBLEM

References:

1. An introduction to classical mechanics by Binoy Bhattacharyya
2. PPT-TR-2_15
3. PPT-Presentation Theories of relativity_p
4. PPT-14_special_rel

EXAMPLE 15.12 A particle of rest mass m_0 moves with velocity v in an inertial frame S . There is another inertial frame S' which is moving with uniform speed u along x -axis. Determine the transformation rule for the momentum four-vectors.

Solution. The transformation rule is given by Eqn (15.160). Hence, we write:

$$\begin{bmatrix} P'_1 \\ P'_2 \\ P'_3 \\ P'_4 \end{bmatrix} = \begin{bmatrix} \gamma & 0 & 0 & i\gamma u/c \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -i\gamma u/c & 0 & 0 & \gamma \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \end{bmatrix} \quad (1)$$

where

$$\gamma = (1 - u^2/c^2)^{-1/2} \quad (2)$$

This yields

$$P'_1 = \gamma \left(P_1 + i\frac{u}{c} P_4 \right) = \gamma \left(mv_x + i\frac{u}{c} \cdot i\frac{E}{c} \right)$$

$$\text{or, } P'_1 = \gamma (mv_x - uE/c^2) \quad (3)$$

where

$$m = m_0 (1 - v^2/c^2)^{-1/2} \quad (4)$$

and $v_x = x$ -component of velocity of the particle in S -frame.

Similarly,

$$P'_2 = P_2 = mv_y, \quad (5)$$

$$P'_3 = P_3 = mv_z \quad (6)$$

$$\text{and } P'_4 = \gamma \left(P_4 - i\frac{u}{c} P_1 \right) = \gamma \left(i\frac{E}{c} - i\frac{u}{c} \cdot mv_x \right)$$

$$\text{or, } P'_4 = \frac{i\gamma}{c} (E - muv_x) \quad (7)$$

Since $P'_4 = i(E'/c)$, Eqn (7) can be written as

$$E' = \gamma(E - muv_x) \quad (8)$$

The inverse transformations are obtained by replacing u by $-u$. Thus,

$$\left. \begin{aligned} P_1 &= \gamma \left(mv'_x + \frac{uE'}{c^2} \right) \\ P_2 &= P'_2 \\ P_3 &= P'_3 \\ E &= \gamma(E' + muv'_x) \end{aligned} \right\} \quad (9)$$